Linear Algebra

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§1 Vector Spaces

§1.1 $\mathbb{R}^n$ and $\mathbb{C}^n$

§1.1.1 Complex Numbers

Definition 1.1 (Complex Numbers)**.** A complex number is an ordered pair (a, b), where $a, b \in \mathbb{R}$, but we will write this as $a + bi$. The set of all complex numbers is denoted by $\mathbb{C}$:

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

Addition and multiplication on $\mathbb{C}$ are defined by

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

If $a \in \mathbb{R}$, we identify $a + 0i$ with the real number a. Thus we can think that $\mathbb{R} \subset \mathbb{C}$.

Example 1.2
Evaluate $(2 + 3i)(4 + 5i)$

$$(8 - 15) + (10 + 12)i = -7 + 22i$$

Proposition 1.3 (Properties of complex arithmetic)
Commutativity:

$$\alpha + \beta = \beta + \alpha \text{ and } \alpha\beta = \beta\alpha \text{ for all } \alpha, \beta \in \mathbb{C}$$

Associativity:

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \lambda) \text{ and } (\alpha\beta)\gamma = \alpha(\beta\gamma) \text{ for all } \alpha, \beta, \gamma \in \mathbb{C}$$

Identities:

$$\lambda + 0 = \lambda \text{ and } \lambda 1 = \lambda \text{ for all } \lambda \in \mathbb{C}$$

Additive Inverse:

$$\forall \alpha \in \mathbb{C} \text{ there exists a unique } \beta \in \mathbb{C} \text{ such that } \alpha + \beta = 0$$

Multiplicative Inverse:

$$\forall \alpha \in \mathbb{C} \text{ with } \alpha \neq 0, \text{ there exists a unique } \beta \in \mathbb{C} \text{ such that } \alpha\beta = 1$$

Distributive Property

$$\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta \text{ for all } \lambda, \alpha, \beta \in \mathbb{C}$$

The properties above are proved using properties of real numbers and definition of complex addition and multiplication.

Example 1.4
Prove that for all $\alpha, \beta \in \mathbb{C}$, $\alpha\beta = \beta\alpha$

Proof. Suppose $\alpha = a + bi$ and $\beta = c + di$. Then,

$$\begin{aligned}\alpha\beta &= (a+bi)(c+di)\\ &= (ac-bd)+(ad+bc)i\\ &= (ca-db)+(da+cb)i\\ &= (c+di)(a+bi)\\ &= \beta\alpha\end{aligned}$$

□

Definition 1.5 (Subtraction and division)**.** Let $\alpha, \beta \in \mathbb{C}$. $-\alpha$ is the unique additive inverse of α so that $\alpha + (-\alpha) = 0$. Subtraction on $\mathbb{C}$ is defined by $\beta - \alpha = \beta + (-\alpha)$.

For $\alpha \neq 0$, let $1/\alpha$ be the unique multiplicative inverse of α so that $\alpha(1/\alpha) = 1$. Division on $\mathbb{C}$ is defined by $\beta/\alpha = \beta(1/\alpha)$

> **Remark 1.6.** Notation: For our purposes, $\mathbb{F}$ stands for either $\mathbb{R}$ or $\mathbb{C}$. This is because $\mathbb{R}$ and $\mathbb{C}$ are both examples of fields.

Definition 1.7 (Scalars)**.** Elements of $\mathbb{F}$ are called scalars. We say scalars to emphasize that an object is a number rather than a vector.

Definition 1.8 (Exponentiation)**.** For $\alpha \in \mathbb{F}$ and m is a positive integer,

$$\alpha^m = \underbrace{\alpha \cdots \alpha}_{m \text{ times}}$$

§1.1.2 Lists

Example 1.9

$\mathbb{R}^2$ and $\mathbb{R}^3$ are examples of lists. The set $\mathbb{R}^2$, which can be thought of as a plane, is the set of all ordered pairs of real numbers:

$$\mathbb{R}^2 = \{(a,b) : a, b \in \mathbb{R}\}$$

The set $\mathbb{R}^3$ can be thought of as ordinary space and is the set of all ordered triples:

$$\mathbb{R}^3 = \{(a,b,c) : a, b, c \in \mathbb{R}\}$$

Definition 1.10 (Lists, lengths)**.** Let $n \in \mathbb{Z}_{\geq 0}$. A list of length n is an ordered collection of n elements (numbers, lists, other abstract entities) separated by commas and surrounded by parentheses:

$$(x_1, x_2, \ldots, x_n)$$

Many mathematicians call a list of length n an n-tuple. A list of length 0 looks like (). We consider such an object to be a list so that certain theorems will not have trivial exceptions. Lists are different from sets because orders and repetitions matter in lists.

Example 1.11

The lists $(3,5) \neq (5,3)$, but the sets $\{3,5\} = \{5,3\}$.
The lists $(4,4) \neq (4,4,4)$ but the sets $\{4,4\} = \{4,4,4\}$.

§1.1.3 $\mathbb{F}^n$

Definition 1.12 ($\mathbb{F}^n$)**.** $\mathbb{F}^n$ is the set of all lists of length n of elements of $\mathbb{F}$:

$$\mathbb{F}^n = \{(x_1, \ldots, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, \ldots, n\}$$

For $(x_1, \ldots, x_n) \in \mathbb{F}^n$ and $j \in \{1, \ldots, n\}$, we say that x_j is the j^{th} coordinate of $(x_1, \ldots, x_n)$.

Example 1.13

$\mathbb{C}^4$ is the set of all lists of four complex numbers:

$$C^4 = \{(z_1, z_2, z_3, z_4) : z_1, z_2, z_3, z_4 \in \mathbb{C}\}$$

Despite not being able to visualize $\mathbb{F}^n$ as a physical object for $n \geq 4$, we can still define algebraic manipulations in $\mathbb{F}^n$ easily.

Definition 1.14 (Addition in $\mathbb{F}^n$)**.** Addition in $\mathbb{F}^n$ is defined by adding corresponding coordinates:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

The mathematics of $\mathbb{F}^n$ becomes cleaner if we use a single variable to denote a list of n numbers without writing out all the coordinates. For example:

Claim 1.15 (Commutativity of addition in $\mathbb{F}^n$) — If $x, y \in \mathbb{F}^n$, then $x + y = y + x$.

Proof.

$$\begin{aligned} x + y &= (x_1, \ldots, x_n) + (y_1, \ldots, y_n) \\ &= (x_1 + y_1, \ldots, x_n + y_n) \\ &= (y_1 + x_1, \ldots, y_n + x_n) \\ &= (y_1, \ldots, y_n) + (x_1, \ldots, x_n) \\ &= y + x \end{aligned}$$

□

Definition 1.16 (0)**.** Let 0 denote the list of length n whose coordinates are all 0:

$$0 = (0, \ldots, 0)$$

Fact 1.17. 0 is the additive identity in $\mathbb{F}^n$.

A typical element of R^2 is a point $x = (x_1, x_2)$. When we think of x as an arrow starting at the origin and ending at (x_1, x_2), we refer to it as a *vector*.

Definition 1.18. For $x \in \mathbb{F}^n$, the additive inverse of x, denoted $-x$, is the vector $-x \in \mathbb{F}^n$ such that

$$x + (-x) = 0$$

i.e. if $x = (x_1, \ldots, x_n)$ then $-x = (-x_1, \ldots, -x_n)$

For a vector $x \in \mathbb{R}^2$, the additive inverse $-x$ is the vector parallel to x and with the same magnitude but pointing in the opposite direction.

Having dealt with addition, we can now deal with multiplication in $\mathbb{F}^n$. However, past experiences have shown that multiplying two elements of $\mathbb{F}^n$ is not very useful for our purposes, so we can instead look toward scalar multiplication, which is multiplying an element of $\mathbb{F}^n$ by an element of $\mathbb{F}$.

Definition 1.19. The product of a number λ and a vector in $\mathbb{F}^n$ is defined as:

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

Where $\lambda \in \mathbb{F}$ and $(x_1, \ldots, x_n) \in \mathbb{F}^n$.

In $\mathbb{R}^2$, we have a nice geometric interpretation (scale by λ in direction depending on sign of λ).

§1.1.4 Digression on Fields

A field is a set containing at least two distinct elements 0 and 1, along with addition and multiplication satisfying all the properties in 1.3. In these notes $\mathbb{F}$ will represent $\mathbb{R}$ or $\mathbb{C}$ but most properties we discuss will hold true for all fields. However, some examples/exercises require that for each positive integer n we have $\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} \neq 0$.

§1.1.5 Exercises 1.1

Exercise 1.20. Suppose a and b are nonzero real numbers. Find real numbers c and d such that

$$1/(a + bi) = c + di$$

We have

$$\begin{aligned}(a + bi)(a - bi) &= a^2 + b^2 \\ \frac{1}{a + bi} &= \frac{a - bi}{a^2 + b^2} \\ &= \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i\end{aligned}$$

So, $c = \frac{a}{a^2+b^2}, d = -\frac{b}{a^2+b^2}$.